The Population Game: A Socially Significant Laboratory Activity

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A game-like activity using dice-like cubes can bring population growth home to all students, scientists, and nonscientists alike, while demonstrating many aspects of probability and uncertainty that are too often ignored in the physics curriculum. The activity can proceed at a variety of levels of sophistication and complication, from a simple demonstration of exponential growth through an elaborate modeling of life expectancy, advanced versus primitive societies, family planning, birth rate, and population momentum. Variations can demonstrate radioactive decay, resource depletion, and the approach of a thermodynamic system to statistical equilibrium.

Humankind needed about 5 million years to reach, in 1825, a population of one billion. We reached our second billion by 1930, our third by 1960, and our *sixth* by 1999. Populations of individual nations such as the United States show similarly surprising growth. A population's annual number of births tends to be proportional to its size, a feature that can be taken as the defining characteristic of exponential growth. Stated differently, populations tend to grow by a constant *percentage* per year rather than by a constant *amount* per year.

Exponential growth can be surprising. All of us, scientists and nonscientists, had better understand these surprises. Al Bartlett[1,2] has taught us the significance of exponential growth, and how to teach it to our students. Bartlett's work has inspired many others, including myself in this paper, to expand on this topic.[3] At least two introductory physics texts for nonscientists present this topic.[4,5]

The Simplest Model (No Death and No Family Planning)

Play in teams of three or four. Each team needs 100 identical dice-like wooden cubes, 1.0 to 1.5 cm on a side, each painted on one face. You can make these from a board 1.0 to 1.5 cm thick that is painted on one side (a different color for each team) by sawing it into ≥ 100 identical cubes. Each team also needs a small bucket, named "Cubeland," graph paper, and a cardboard box.

Students could begin by investigating simple probabilistic aspects of the cubes. Toss, say, 10 cubes and count the number and the fraction that come up painted. Repeat several times and graph the results. Do the same with 50 cubes and 100 cubes. What do students notice about the fraction coming up painted in the three cases (it "should" get closer to 1/6)?

Begin the game with all cubes in the box. Remove one cube, put it into Cubeland, and toss it onto a level surface. If it comes up painted, it "reproduced" that year, and the team should add one more cube to Cubeland. Note that Cubelanders procreate asexually, like bacteria.[6] Put the one or two inhabitants back into the bucket and toss again. Add zero, one, or two inhabitants to Cubeland, depending on the outcome of the toss. Continue: Each toss represents one year in Cubeland history, and the number of painted sides showing on each toss is the number of "babies" that year. Graph the population N versus the year, beginning with $N_0 = 1$ in year 0. Cubeland's "carrying capacity" is arbitrarily deemed to be $N = 100$. The object of the game is "sustainability," which we arbi-

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trarily define as survival of the population for 50 years without either dying out ($N = 0$) or exceeding the carrying capacity.

As the game proceeds, you could ask students: How long do you think it will be before Cubeland is full? Predict the shape of the graph. Will the number of new offspring be approximately the same every year? What *will* be approximately the same every year? (This is a key point: The *percentage increase* is approximately the same each year). What is the expected (or average) percentage increase in population each year (1/6 = 16.7%)? Is this the actual percentage increase in every year?

Exponential growth is identifiable in either of two equivalent ways: (1) The percentage increase per unit of time is unchanging. (2) The doubling time (the time for any population N to grow to $2N$) is unchanging. Thus, the preceding questions lead to the conclusion that the expected (or average) growth of Cubeland is exponential.

Activities: Estimate the doubling time from the graph and check whether it is indeed unchanging. Where and why do deviations occur? Compare the observed annual percentage increase with the prediction for this game that $P \approx 75/T$ (see the following paragraphs) and also with $P \approx 1/6$.

Many questions can now be pursued. For example: Suppose some enterprising Cubelander discovers an entire empty bucket into which the population can continue expanding. How long will it be before this bucket, too, is full? Exhibit a graph of the population explosion[7] and ask students if, judging from doubling times, it really is exponential. As the statistics quoted in the opening paragraph show, it was faster than exponential during most of this time. More precisely, it was faster than exponential until 1970, after which it was slower than exponential.

For those who want a more detailed analysis: Since each cube has a probability of 1/6 of reproducing in any given year, the expected population in the kth year is

$$\langle N_k\rangle = \langle N_{k-1}\rangle + (1/6)\langle N_{k-1}\rangle = (7/6)\langle N_{k-1}\rangle$$
$$= \dots = (7/6)^k \langle N_0\rangle = (7/6)^k.$$

Time is in the exponent, which is why we call it "exponential growth." Solving $(7/6)^k = 2$, we find a doubling time of $T = 4.497 \approx 4.5$ yr. Thus, $PT = 16.7 \times 4.5 = 75$, larger than the prediction for continuous (not stepwise) exponential growth $PT \approx 70$.[5,8,9]

Inform students that the theoretical doubling time is about 4.5 years, or ask them to calculate and/or find it experimentally. Ask each team to make a table showing the expected population every 4.5 years (see Table I). Apparently, Cubeland reaches overpopulation ($N = 100$) in about 30 years. Individual student graphs, and the class average graph, could be compared with the table. Using $PT \approx 70$, ask the class to find the doubling times for China ($P = 0.6\%\ y^{-1}$), the United States ($1.2\%\ y^{-1}$), Pakistan ($2.8\%\ y^{-1}$), and the European Union ($0.1\%\ y^{-1}$).

Table I. The expected population of Cubeland.

k = years	doublings	$\langle N_k\rangle = (7/6)^k$
0	0	1
4.5	1	2
9	2	4
13.5	3	8
18	4	16
22.5	5	32
27	6	64
31.5	7	128
36	8	256
40.5	9	512

Life Expectancy

The previous model is highly unrealistic for human populations, because death is not included and so overpopulation is inevitable. Once students have played the game as described, they are ready to experiment with a finite life expectancy.

The game can model life expectancy by simply specifying a fixed lifetime for all inhabitants, using the recorded data to remove the necessary number each year, and otherwise playing the game as described. It is helpful to keep a table of births B, deaths D, and population increase Δ each year. Suppose, for example, that the lifetime is 10 years. Then one cube should be removed from Cubeland at the end of the 10th year (because one cube was "born" in year zero);

Table II. The probability P_k of a Cubelander having its first baby in its *k*th year and its cumulative probability of having its first baby during its first *k* years.

k	P_k	$\Sigma_{n=1}^{k} P_n$
1	0.167	0.167
2	0.139	0.306
3	0.116	0.421
4	0.0964	0.518
5	0.0803	0.598
6	0.670	0.665

for any year $k > 10$, the number removed should be the same as the number born in year $k-10$.

What is a reasonable life expectancy for Cubelanders? For survival of a population, the life expectancy should be at least comparable to the "child"-bearing age. The probability of a particular cube having its first "baby" in its *k*th year is

$$P_k = \text{Prob (unpainted at the first } k\text{-1 throws)} \times \text{Prob (painted at } k\text{th throw)} = (5/6)^{k-1} \times (1/6)\ .$$

Table II shows this probability P_k of a single cube having its first baby in its *k*th year, and the cumulative probability that a single cube will have its first baby sometime *during* its first *k* years. Apparently, the median age of a cube at the time of its first baby is a little less than 4 years. You could just provide this information to students, or ask them to calculate and/or find it experimentally.

Thus, a primitive Cubeland would have a life expectancy of 4 to 6 years, just a little larger than the childbearing age. Different teams should try life expectancies of say 4, 6, 8, 10, 12, and 14 years. Among nations, Japan's 82-year average life expectancy — three or four times the human childbearing age — is the world's longest. Average life expectancy in the most advanced nations has risen linearly for 160 years and could reach 100 within six decades.[10]

Cubeland societies with a four-year life expectancy will probably become extinct ($N = 0$) during the first few years. There is also a high probability of extinction with lifetimes of 6 to 8 years. There is a lesson here, of course: Populations that are both small and very primitive are highly susceptible to extinction. These teams should then start over with a new graph and a longer assigned lifetime. Teams with lifetimes of 12 to 14 years will probably reach overpopulation well before 50 years, mirroring the fact that the human population explosion is a consequence of increased life expectancy arising from medical advances. This suggests that advanced societies cannot survive without family planning.

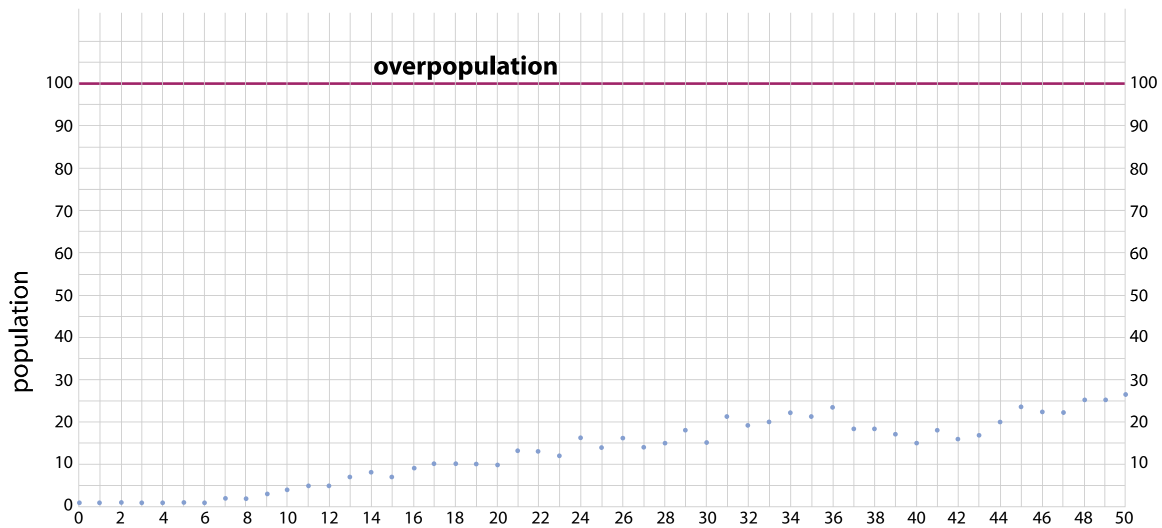
Family Planning

After students have played the game with finite lifetimes, they are ready to add family planning. To model universal one-child family planning (all inhabitants have at most one child), let the game proceed with a finite lifetime and no family planning until the population has reached a certain size, and then institute family planning as illustrated in the following example.

Choose a lifetime such as 14 years, corresponding to an advanced nation, and play the game with no family planning until the population passes, say, $N = 60$. At this point, a Cubeland physicist notices the population problem and recommends a one-child family-planning policy beginning next year. On the next throw, all the cubes that come up painted should be put into a special "no-further-babies" zone outside of the bucket (although these are still counted as citizens of Cubeland). In order not to make mistakes, the team should keep careful track of B, D, and Δ each year. It is also a good idea to specifically add B_k cubes to the bucket in the *k*th year, and then separately remove D_k ($= B_{k-14}$) cubes.

There is one additional proviso: Deaths must come *proportionally* from *both* the no-babies zone and from the bucket. For example, if there are about twice as many in the bucket as in the no-babies zone, then about two-thirds of the deaths should come from the bucket and one-third from the no-babies zone.

Students can observe "population momentum" in this model: Rapid growth continues for a few years after family planning has started. This can carry Cubeland to overpopulation even when family planning starts at, say, $N = 60$. But if planning starts at $N = 40$, for example, the graph will probably continue rising but at a reduced rate, and then level off and de-



B	1	0	0	0	0	0	1	1	0	1	1	1	1	3	1	0	3	2	1	3	1	3	3	1	5	1
D	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	1	1	1	1	3	1	0	3	2	1	3
Δ	1	0	0	0	0	0	0	1	0	1	1	1	0	2	1	-1	2	1	0	0	0	3	0	-1	4	-2

B	3	1	4	4	2	7	1	2	6	3	4	2	1	1	4	6	2	3	4	4	3	6	5	3	5
D	1	3	3	1	5	1	3	1	4	4	2	7	1	2	6	3	4	2	1	1	4	6	2	3	4
Δ	2	-2	1	3	-3	6	-2	1	2	-1	2	-5	0	-1	-2	3	-2	1	3	3	-1	0	3	0	1

Fig. 1. Fifty-year population history of Cubeland for a primitive society having individual lifetimes of only 6 years, slightly above the childbearing age of about 4 years, with no family planning.

cline in a bell-shaped curve.

Two-child family planning can be modeled by using a second bucket (for cubes that have had one baby) in addition to the no-further-babies zone (for cubes that have had two). I'll leave the details to the reader.

Students can model various real-world scenarios: (1) A less-developed nation in which life expectancy is low and there is no family planning. (2) An initially less-developed nation that, after a certain number of years, becomes more developed (longer life expectancy) but still with no family planning. (3) A nation that proceeds through the "demographic transition" — less-developed period of short life expectancy and no family planning, followed by a development period of longer life expectancy and no family planning, followed by a developed period of long life expectancy and family planning.[11] Different teams can experiment with different life expectancy and family-planning assumptions.

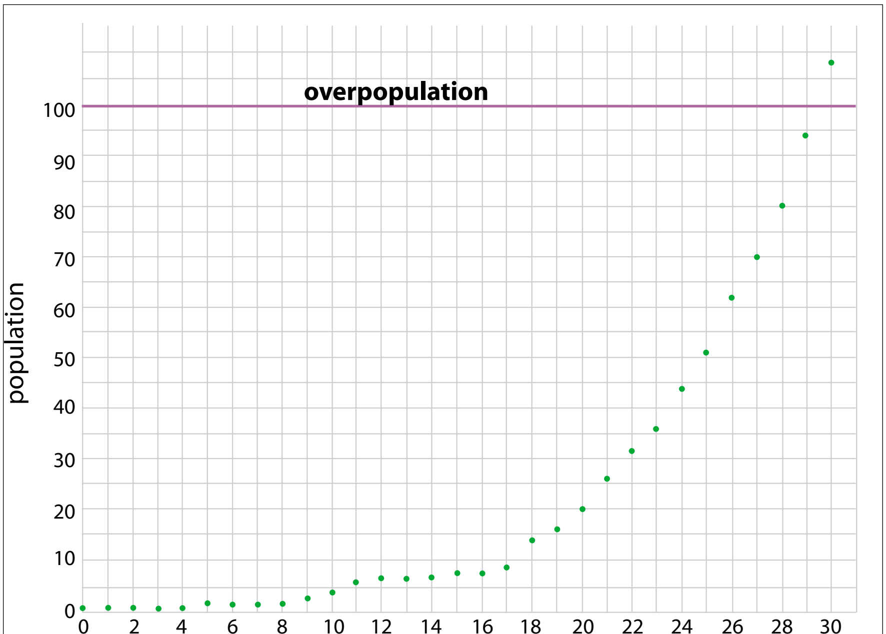
Teams can compete to attain a specific sustainable population goal. Each team should choose a particular set of assumptions and play the game with those fixed assumptions. The team whose population levels off closest to the pre-assigned goal wins.

Some teams could experiment with cubes painted on *two* sides (paint the board on both sides before sawing it into cubes) to mirror the effect of a higher annual per capita birthrate. The game could be completed faster and with more variations on a computer, using a random number generator. A computer could vary the longevity, family-planning options, maximum population, number of years of data, and especially the birthrate, which is set at 16.7% babies per cube per year in the Cubeland model. But my guess is that this would not be as much fun as the hands-on version.

Classroom Trials

The simplest models (the life-expectancy model and the life-expectancy-plus-one-child-family-planning model) were tested at two workshops in Guilin, China. There were about 25 people in each workshop, divided into seven teams. The workshop attendees were graduate students in physics education at Guangxi Normal University in Guilin and university physics professors from all over China. Each workshop spent about four hours playing all three versions of the game and discussing the pedagogy and the societal implications.

Figures 1–3 show typical results. Each figure is a graph of Cubeland's population during 50 "years" of play. Births are tabulated in row "B" below the graph, deaths are tabulated in row "D," and the population



B	1	0	0	0	0	1	0	0	0	1	1	2	1	0	1	1	0	1	5	3	4	6	6	5	9	9	12	8	11	15	11
D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	1	2	1	0	1	1	0
Δ	1	0	0	0	0	1	0	0	0	1	1	2	1	0	0	1	0	1	5	2	4	6	6	4	8	7	11	8	10	14	14

Fig. 2. Population history for an advanced society having individual lifetimes of 14 years and universal one-child family planning beginning when the population passes 60.

increase each year is tabulated in row "Δ."

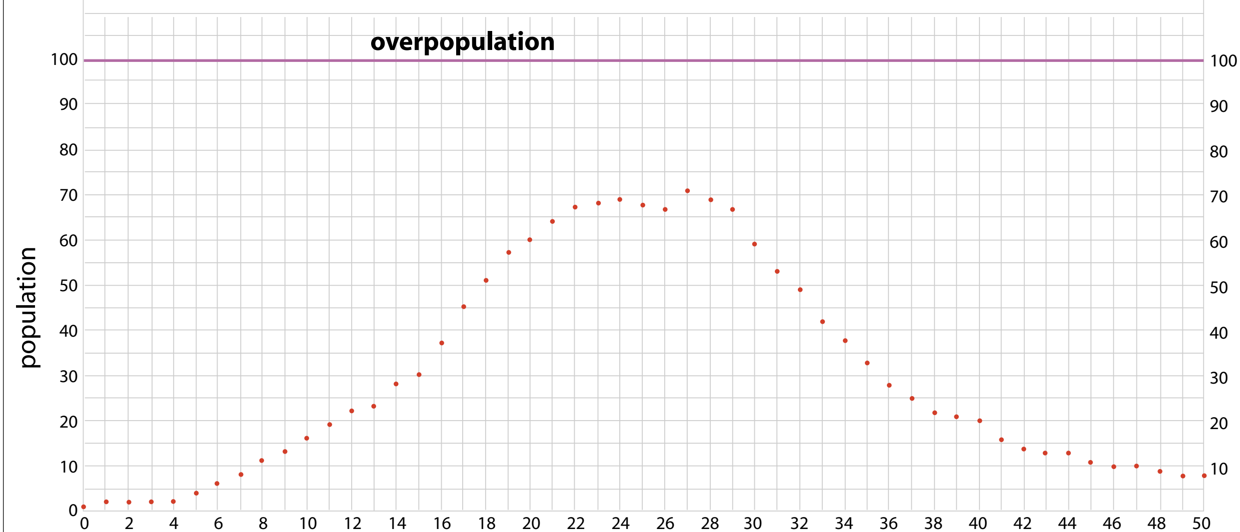
The conditions used for Fig. 1 are a lifetime of six years and no family planning. A six-year lifetime is barely above the childbearing age of about four years and corresponds to a primitive society. In the workshop, such societies either became extinct or grew very slowly. In Fig. 1, the population grows very slowly, mirroring the slow growth of the human population from about 5 million years ago until the advent of more modern sanitation and medicine around 1400 A.D. The population of Fig. 1 reaches the 50-year mark with a population of only 26, well below overpopulation. At least during this time, family planning is not needed. Note that 50 years is 12 times the average childbearing age, and so corresponds to some three centuries of human history.

The conditions for Fig. 2 are a lifetime of 14 years and one-child family planning beginning when the population passes 60. The 14-year lifetime is about four times the childbearing age and corresponds to an advanced society such as Japan. The necessity of family planning in such a society can be seen in Fig. 2, which is clearly on the path to overpopulation by the time $N = 60$. The population first exceeds 60 in year 26, and the one-child policy is instituted. But the new policy starts too late: By year 30, the population exceeds 100. We see here the effect of population momentum. The preponderance of younger individuals (those cubes that have not yet been placed into the no-further-babies zone) implies large population increases even long after the one-child policy begins.

Figure 3, on the other hand, corresponds to a more rational advanced society. Again, the lifetime is 14 years, but the one-child policy begins when the population passes 30. This level is reached in year 15. Then, the slope of the graph begins to decrease, but population momentum maintains a population increase until about year 27, when the graph finally turns around. The society sustains itself for 50 years, but by this time the population has dropped to the dangerously low level of eight. Apparently, this society would have done well to relax its one-child policy, perhaps to a two-child policy, around year 32.

Other Phenomena

Radioactive decay:[12] Start with *all* the cubes in the bucket, toss them, and *remove* the ones that come up painted because they "decayed" during that year. Put the undecayed cubes back in the bucket and toss them, again removing the decayed cubes, etc. At what "time" (number of throws) is the "radioactive sample" (cubium?) down to one-half of its original size? Predict the time at which the sample will be down to one-fourth. To one-eighth. Predict the time until the sample is entirely decayed. Compare the results of different teams. Try two species mixed together, one with a single side painted and the other with two sides painted.



B	1	1	0	0	0	2	2	2	3	2	3	3	3	1	6	3	8	8	6	8	5	6	6	3	4	2
D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	2	2	2	3	2	3	3
Δ	1	1	0	0	0	2	2	2	3	2	3	3	3	1	5	2	8	8	6	6	3	4	3	1	1	-1

B	2	5	4	1	0	2	2	1	1	1	1	0	1	1	1	1	2	0	0	0	1	1	0	0	1
D	3	1	6	3	8	8	6	8	5	6	6	3	4	2	2	5	4	1	0	2	2	1	1	1	1
Δ	-1	4	-2	-2	-8	-6	-4	-7	-4	-5	-5	-3	-3	-1	-1	-4	-2	-1	0	-2	-1	0	-1	-1	0

Fig. 3. Population history for a society that is identical to that of Fig. 2, except that the one-child policy is adopted when the population passes 30.

Consumption of a nonrenewable resource: Each team needs three buckets. Buckets A and B contain the resource, "cube ore," initially divided equally between the two buckets. Bucket C, initially empty, represents the cumulative production of cube ore. Place 50 cubes in A and 50 in B. Remove one cube from A and place it in C, representing the first successful mining of cube ore. Toss the cube from C. After one or more tosses (years), it comes up painted, representing further mining: Move one more cube from A to C. Toss both of the cubes in C and move 0, 1, or 2 cubes from A to C depending on the number that come up painted. Continue, graphing the annual production, the cumulative production C, and the remaining resource A + B until A is depleted. If a few additional cubes are needed on the last toss, take them from B.

The total resource (A + B) is now half depleted and production becomes more difficult (typical for nonrenewable resources). To represent this turnaround point, begin tossing the cubes *in bucket B* because the production process is now resource limited (bucket B) rather than production-efficiency limited (bucket C). Oil has passed this turnaround point in the United States but is still approaching it worldwide. As usual, the cubes that come up painted have been "mined" and are moved from B to C. Continue mining bucket B, continuing all three graphs.

Irreversibility and the approach to statistical equilibrium: Use two buckets, A and B. Place all 100 cubes in A. The two buckets represent the left and right halves of a box that contains "cubium" gas molecules. Initially, the gas is entirely in the left half, kept there by an impermeable membrane. At $t = 0$, the membrane is punctured and cubium begins diffusing into B. Toss the cubes. Cubes coming up painted are deemed to have diffused from A to B; put them into bucket B. Now toss both buckets. Cubes from A coming up painted have diffused to B, while cubes from B coming up painted have diffused to A. Continue, graphing the two populations, and observe the evolution to equilibrium. This model is basically Paul Ehrenfest's "double-urn" model of the approach to equilibrium.[13]

The reader can probably dream up additional ways of using the population cubes.

Acknowledgment

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4. Paul Hewitt, *Conceptual Physics,* 9th ed. (Addison-Wesley, San Francisco, 2002), pp. 755–761.
5. Art Hobson, *Physics: Concepts and Connections,* 3rd edition (Prentice Hall, Upper Saddle River, NJ, 2003), pp. 186–191.
6. For a sexually reproducing model, use 50 unpainted cubes ("males") and 50 painted on one side ("females"). Begin with one male and one female, and the other 98 in the box and proceed as described in this section, with the proviso that each new "baby" must be chosen *at random* from the single pool of males and females. The expected population in the *k*th year is now $<N_k> = 2\,(13/12)^k$, so the game proceeds more slowly. To speed it up, use female cubes that are painted on both sides.
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13. J.A. Wheeler, "Time Today," in *Physical Origins of Time Asymmetry*," edited by J.J. Halliwell, J. Perex-Mercader, and W. H. Zurek (Cambridge University Press, Cambridge, UK, 1994), p. 1.

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